Engineering Notes

Canonical Solution of Two-Point Boundary-Value Problems

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DOI: 10.2514/1.51098

I. Introduction

TWO-POINT boundary-value problems (TPBVPs) have been widely studied in the field of mathematics, mechanics, and control. Widely used methods for nonlinear TPBVPs include shooting, interpolation, finite difference, and many of their variants. In aerospace engineering, a recent focus on spacecraft formation flying raised interest in TPBVPs related to the relative motion of spacecraft. Jiang et al. [1] studied a TPBVP of spacecraft formation flying in an unperturbed elliptical reference orbit. They reformed the problem into a relative Lambert's problem and pointed out that it can be solved similarly to the classical Lambert problem. Approximate analytic solutions and numerical solutions were obtained. However, significant perturbations, such as the J_2 term, were not considered in their work. Guibout and Scheeres [2] and Guibout [3] studied the reconfiguration problem of spacecraft formations in a general Hamiltonian dynamic environment. They solved the Hamilton–Jacobi equation using a series expansion and numerical integration to obtain an analytic approximation of the generating functions. Based on the theory of canonical transformations, the relative TPBVPs were solved using partial derivatives of the generating functions. In [2,3], all nondissipative perturbations could be considered. However, acquiring a good approximation of the generating functions may require extensive computations.

The research effort of this Note is inspired by the generating function method to solve TPBVPs in [2,3]. It is motivated by the following observations. For an integrable Hamiltonian system, the generating functions can be obtained analytically; for a Hamiltonian system that is not analytically integrable, a perturbation theory can be used if the main part of the Hamiltonian is integrable. By such observations, TPBVPs for those Hamiltonian systems that have a main integrable part may be solved more efficiently.

In this Note, a canonical method is developed to solve TPBVPs for Hamiltonian systems with a main integrable part. The method uses the canonical solution of Hamilton–Jacobi equations by separation of variables and canonical perturbation theory; thus, it can be described as canonical. The canonical method takes all perturbations of Hamiltonian into account uniformly, and it is computationally efficient. An application to satellite formation flying with a circular reference orbit is derived, and case studies are presented.

Received 9 June 2010; revision received 27 August 2010; accepted for publication 28 August 2010.

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II. Canonical Method for Two-Point Boundary-Value Problems of Hamiltonian Systems

For a Hamiltonian dynamical system with the Hamiltonian, $H(\boldsymbol{q}, \boldsymbol{p}, t)$, and the canonical equations [4,5],

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} \tag{1}$$

the TPBVP is to find the boundary values of the generalized momenta, $\boldsymbol{p}(t_0) = \boldsymbol{p}_0$ and $\boldsymbol{p}(t_f) = \boldsymbol{p}_f$, given the initial time t_0, the final time t_f, and the generalized coordinates $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\boldsymbol{q}(t_f) = \boldsymbol{q}_f$. The boundary condition stated here is a simple one. It is often encountered in orbital dynamics problems. This simple formulation of the TPBVP is addressed in developing the canonical method. However, the main approach developed in this Note does not lose its generality for other kinds of boundary conditions, although different boundary constraints may introduce additional algebraic equations that may not be easy to solve.

General solutions to the previous problem are not going to be obtained in this Note. Practically, the problem is restricted to a class of the Hamiltonian dynamical system. Typically, in a physical problem that cannot be solved directly, the Hamiltonian differs only slightly from the Hamiltonian for a problem that can be solved rigorously [4]. This is true for the orbital dynamics problems of interest in this Note. For these Hamiltonian systems, the Hamiltonian function usually consists of two parts: the main Hamiltonian, for which the Hamilton–Jacobi equation can be solved by separation of variables, and a relatively small perturbation Hamiltonian. Thus, the complete motion can be obtained by canonical perturbation theory. More details about the theory of Hamilton mechanics can be found elsewhere [4,5]. However, the interest here is not in solving Hamiltonian systems but rather in solving TPBVPs in a Hamiltonian dynamic environment. In this section, the problem is addressed step by step.

A. Solving Two-Point Boundary-Value Problems for Separable Hamilton Systems

If the Hamiltonian is separable or, more specifically, if the Hamilton–Jacobi equation can be solved by separation of variables, it is simple to solve TPBVPs using the results of the Hamilton–Jacobi theory.

For Hamiltonian systems with a separable Hamiltonian, $H_0(\boldsymbol{q}, \boldsymbol{p}, t)$, let $S(\boldsymbol{q}, \boldsymbol{P}, t)$ represent the Hamilton's principal function, i.e., a complete solution of the Hamilton–Jacobi equation:

$$H_0\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{2}$$

The canonical transformation relationships are

$$\boldsymbol{p} = \frac{\partial S(\boldsymbol{q}, \boldsymbol{P}, t)}{\partial \boldsymbol{q}}, \qquad \boldsymbol{Q} = \frac{\partial S(\boldsymbol{q}, \boldsymbol{P}, t)}{\partial \boldsymbol{P}} \tag{3}$$

where the new Hamiltonian $K \equiv 0$ and $(\boldsymbol{Q}, \boldsymbol{P})$ are the canonical constants.

Because Eq. (2) can be solved by the separation of variables, the analytic solution of $S(\boldsymbol{q}, \boldsymbol{P}, t)$ is ready for use. Thus, from Eq. (3), the solutions of motion can be obtained:

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q}, \boldsymbol{P}, t), \qquad \boldsymbol{p} = \boldsymbol{p}(\boldsymbol{Q}, \boldsymbol{P}, t) \tag{4}$$

If $\boldsymbol{q}_0$ and $\boldsymbol{q}_f$ are given, according to the second relationship of Eq. (3),

$$\boldsymbol{Q} = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{q}_0, \boldsymbol{P}, t_0) = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{q}_f, \boldsymbol{P}, t_f) \tag{5}$$

$\boldsymbol{P}$ can be solved. Then, $\boldsymbol{p}_0$ and $\boldsymbol{p}_f$ are solved using the first relationship of Eq. (3).

B. Solving Two-Point Boundary-Value Problems for Perturbed Hamiltonian Systems

When there is a perturbation part, $H_1(\boldsymbol{q},\boldsymbol{p},t)$, in addition to the main separable Hamiltonian, $H_0(\boldsymbol{q},\boldsymbol{p},t)$, the full Hamiltonian is $H(\boldsymbol{q},\boldsymbol{p},t) = H_0(\boldsymbol{q},\boldsymbol{p},t) + H_1(\boldsymbol{q},\boldsymbol{p},t)$. From canonical perturbation theory [4], the solution of systems with a perturbed Hamiltonian $H(\boldsymbol{q},\boldsymbol{p},t)$ has the same form as the solution of a system with an unperturbed Hamiltonian, $H_0(\boldsymbol{q},\boldsymbol{p},t)$, i.e., Eq. (4). However, $\boldsymbol{Q}$ and $\boldsymbol{P}$ are no longer constants; they satisfy the following perturbation differential equations:

$$\dot{\boldsymbol{Q}} = \frac{\partial H_1}{\partial \boldsymbol{P}}(\boldsymbol{Q},\boldsymbol{P},t), \qquad \dot{\boldsymbol{P}} = -\frac{\partial H_1}{\partial \boldsymbol{Q}}(\boldsymbol{Q},\boldsymbol{P},t) \tag{6}$$

where $H_1(\boldsymbol{Q},\boldsymbol{P},t)$ is the perturbation Hamiltonian in the form of $(\boldsymbol{Q},\boldsymbol{P})$ after substituting Eq. (4) into $H_1(\boldsymbol{q},\boldsymbol{p},t)$.

The second relationship of Eq. (3) provides the boundary conditions for Eq. (6):

$$\boldsymbol{Q}_0 = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{q}_0,\boldsymbol{P}_0,t_0), \qquad \boldsymbol{Q}_f = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{q}_f,\boldsymbol{P}_f,t_f) \tag{7}$$

If $\boldsymbol{P}_0$ and $\boldsymbol{P}_f$ are obtained, then $\boldsymbol{p}_0$ and $\boldsymbol{p}_f$ can be determined using the first relationship of Eq. (3), and the TPBVP is solved.

Thus, the original TPBVP is converted to a new one defined by Eqs. (6) and (7). The new TPBVP does not appear to be simpler; however, recalling the assumption that the perturbation Hamiltonian is relatively small, ways can be found to solve this new TPBVP conveniently by linearization.

C. Linearization of Canonical Perturbation Equations and Boundary Conditions

Under the assumption that $H_1(\boldsymbol{q},\boldsymbol{p},t)$ is very small compared with $H_0(\boldsymbol{q},\boldsymbol{p},t)$, it is reasonable to accept that the differences between $(\boldsymbol{Q},\boldsymbol{P})$, variable canonical constants of the perturbed problem, and $(\bar{\boldsymbol{Q}},\bar{\boldsymbol{P}})$, canonical constants of the unperturbed problem with the Hamiltonian $H_0(\boldsymbol{q},\boldsymbol{p},t)$, are relatively small. Thus, the new TPBVP can be linearized.

From Sec. II.A, the unperturbed canonical constants $(\bar{\boldsymbol{Q}},\bar{\boldsymbol{P}})$ are obtained, given $\boldsymbol{q}_0$, $\boldsymbol{q}_f$, t_0, and t_f. $(\bar{\boldsymbol{Q}},\bar{\boldsymbol{P}})$ is a first approximate solution of the perturbation equations, i.e., Eq. (6) with the boundary conditions given in Eq. (7). If the perturbation equations are linearized around the approximate solution, $\overline{\boldsymbol{Q}(t)} = \bar{\boldsymbol{Q}}$ and $\overline{\boldsymbol{P}(t)} = \bar{\boldsymbol{P}}$, the linearized differential equations are

$$\begin{bmatrix} \Delta\dot{\boldsymbol{Q}} \\ \Delta\dot{\boldsymbol{P}} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H_1}{\partial \boldsymbol{P}\partial \boldsymbol{Q}} & \frac{\partial^2 H_1}{\partial \boldsymbol{P}^2} \\ -\frac{\partial^2 H_1}{\partial \boldsymbol{Q}^2} & -\frac{\partial^2 H_1}{\partial \boldsymbol{Q}\partial \boldsymbol{P}} \end{bmatrix} (\overline{\boldsymbol{Q}(t)},\overline{\boldsymbol{P}(t)},t) \begin{bmatrix} \Delta\boldsymbol{Q} \\ \Delta\boldsymbol{P} \end{bmatrix} + \begin{bmatrix} \frac{\partial H_1}{\partial \boldsymbol{P}} \\ -\frac{\partial H_1}{\partial \boldsymbol{Q}} \end{bmatrix} (\overline{\boldsymbol{Q}(t)},\overline{\boldsymbol{P}(t)},t) - \begin{bmatrix} \dot{\overline{\boldsymbol{Q}(t)}} \\ \dot{\overline{\boldsymbol{P}(t)}} \end{bmatrix} \tag{8}$$

where

$$\begin{bmatrix} \Delta\boldsymbol{Q}(t) \\ \Delta\boldsymbol{P}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}(t) \\ \boldsymbol{P}(t) \end{bmatrix} - \begin{bmatrix} \overline{\boldsymbol{Q}(t)} \\ \overline{\boldsymbol{P}(t)} \end{bmatrix}$$

Similarly, the new boundary conditions in Eq. (7) are linearized around $\overline{\boldsymbol{Q}(t)}$ and $\boldsymbol{P}(t)$:

$$\begin{bmatrix} I & -\frac{\partial^2 S}{\partial \boldsymbol{P}^2}(\boldsymbol{q}_0,\overline{\boldsymbol{P}(t_0)},t_0) \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{Q}(t_0) \\ \Delta\boldsymbol{P}(t_0) \end{bmatrix} = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{q}_0,\overline{\boldsymbol{P}(t_0)},t_0) - \overline{\boldsymbol{Q}(t_0)}$$
$$\begin{bmatrix} I & -\frac{\partial^2 S}{\partial \boldsymbol{P}^2}(\boldsymbol{q}_f,\overline{\boldsymbol{P}(t_f)},t_f) \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{Q}(t_f) \\ \Delta\boldsymbol{P}(t_f) \end{bmatrix} = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{q}_f,\overline{\boldsymbol{P}(t_f)},t_f) - \overline{\boldsymbol{Q}(t_f)} \tag{9}$$

Equations (8) and (9) describe a TPBVP of a linear differential system with linear boundary conditions. It is easy to solve these kinds of linear TPBVPs. The procedure to solve this problem can be found in [6].

D. Obtaining High-Order Precision by Iterations

The solution of the linearized problem in Sec. II.C gives the first-order approximation of the TPBVP. To improve the solution, the linearization procedure in Sec. II.C is repeated iteratively by letting $\overline{\boldsymbol{Q}(t)} = \boldsymbol{Q}(t)$ and $\overline{\boldsymbol{P}(t)} = \boldsymbol{P}(t)$ to update the approximate solution. Theoretically, any desired precision can be achieved by iterations. The idea of iteration comes from [4]. It is explained in the following.

To be concise, let $\boldsymbol{r} = (\boldsymbol{Q},\boldsymbol{P})$, then the perturbation differential equations of Eqs. (7) become

$$\dot{\boldsymbol{r}} = \mathbf{J}\frac{\partial H_1}{\partial \boldsymbol{r}}(\boldsymbol{r},t) \tag{10}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Let $\boldsymbol{r}_0 = (\bar{\boldsymbol{Q}},\bar{\boldsymbol{P}})$, which are the canonical constants of the main integrable Hamiltonian, $H_0(\boldsymbol{q},\boldsymbol{p},t)$. By Taylor expansion at $\boldsymbol{r}_0$, the perturbation Hamiltonian is

$$H_1(\boldsymbol{r},t) = H_1(\boldsymbol{r}_0,t) + \left.\frac{\partial H_1}{\partial \boldsymbol{r}}\right|_0 (\boldsymbol{r}-\boldsymbol{r}_0) + \left.\frac{\partial^2 H_1}{\partial \boldsymbol{r}^2}\right|_0 \frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}_0)^2 + \left.\frac{\partial^3 H_1}{\partial \boldsymbol{r}^3}\right|_0 \frac{1}{6}(\boldsymbol{r}-\boldsymbol{r}_0)^3 + o(|\boldsymbol{r}-\boldsymbol{r}_0|^3) \tag{11}$$

If the $o(|\boldsymbol{r}-\boldsymbol{r}_0|^2)$ terms in the perturbation Hamiltonian are dropped and then substituted into Eq. (10), the linearized perturbation equation is obtained:

$$\dot{\boldsymbol{r}} = \mathbf{J}\frac{\partial H_1}{\partial \boldsymbol{r}}(\boldsymbol{r}_0,t) + \frac{\partial^2 H_1}{\partial \boldsymbol{r}^2}(\boldsymbol{r}_0,t)(\boldsymbol{r}-\boldsymbol{r}_0) \tag{12}$$

which is the same as Eqs. (8). By iteration, substitute $\boldsymbol{r}_1$, the solution of Eq. (12), for $\boldsymbol{r}_0$ in Eq. (12):

$$\dot{\boldsymbol{r}} = \mathbf{J}\frac{\partial H_1}{\partial \boldsymbol{r}}(\boldsymbol{r}_1,t) + \frac{\partial^2 H_1}{\partial \boldsymbol{r}^2}(\boldsymbol{r}_1,t)(\boldsymbol{r}-\boldsymbol{r}_1) \tag{13}$$

Express the right side of Eq. (13) with the Taylor expansion at $\boldsymbol{r}_0$:

$$\begin{aligned}\dot{\boldsymbol{r}} &= \mathbf{J}\frac{\partial H_1}{\partial \boldsymbol{r}}(\boldsymbol{r}_0,t) + \mathbf{J}\frac{\partial^2 H_1}{\partial \boldsymbol{r}^2}(\boldsymbol{r}_0,t)(\boldsymbol{r}_1-\boldsymbol{r}_0) + \mathbf{J}\frac{\partial^3 H_1}{\partial \boldsymbol{r}^3}(\boldsymbol{r}_0,t)\frac{1}{2}(\boldsymbol{r}_1-\boldsymbol{r}_0)^2 + o(|\boldsymbol{r}_1-\boldsymbol{r}_0|^2) \\ &\quad + \left[\frac{\partial^2 H_1}{\partial \boldsymbol{r}^2}(\boldsymbol{r}_0,t) + \frac{\partial^3 H_1}{\partial \boldsymbol{r}^3}(\boldsymbol{r}_0,t)(\boldsymbol{r}_1-\boldsymbol{r}_0) + o(|\boldsymbol{r}_1-\boldsymbol{r}_0|)\right](\boldsymbol{r}-\boldsymbol{r}_1)\end{aligned} \tag{14}$$

Thus,

$$\begin{aligned}\dot{\boldsymbol{r}} &= \mathbf{J}\frac{\partial H_1}{\partial \boldsymbol{r}}(\boldsymbol{r}_0,t) + \mathbf{J}\frac{\partial^2 H_1}{\partial \boldsymbol{r}^2}(\boldsymbol{r}_0,t)(\boldsymbol{r}-\boldsymbol{r}_0) \\ &\quad + \mathbf{J}\frac{\partial^3 H_1}{\partial \boldsymbol{r}^3}(\boldsymbol{r}_0,t)\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}_0)^2 - \mathbf{J}\frac{\partial^3 H_1}{\partial \boldsymbol{r}^3}(\boldsymbol{r}_0,t)\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}_1)^2 \\ &\quad + o(|\boldsymbol{r}_1-\boldsymbol{r}_0|^2) + o(|\boldsymbol{r}_1-\boldsymbol{r}_0|)(\boldsymbol{r}-\boldsymbol{r}_1)\end{aligned} \tag{15}$$

Because $\boldsymbol{r}_1$ is the low-order approximation of $\boldsymbol{r}$, the last three right-hand terms of Eq. (15) are all $o(|\boldsymbol{r}-\boldsymbol{r}_0|^2)$ terms. Therefore, Eq. (15) is equivalent to the second-order approximation of the perturbation differential equations (i.e., the perturbation Hamiltonian approximation to the third order). This implies that, by repeatedly using the first-order approximation equation, second-order precision can be achieved. When this procedure is done iteratively, any precision can be theoretically achieved.

E. Remarks on the Canonical Method

It is worth summarizing the advantages taken in the canonical method. The motion must be described in the Hamiltonian canonical form; the Hamiltonian must be integrable or have a main integrable part, and the nonintegrable perturbation Hamiltonian must be small. These conditions enable the use of the Hamilton–Jacobi theory and the canonical perturbation theory. The form of the Hamiltonian principal function that relates the boundary values can be obtained without considering perturbations at first. Then, perturbations are taken into account by variation of canonical constants. Finally, for a small perturbation Hamiltonian, the variation of canonical constants can be solved by linearization, which is the key step to solve TPBVPs for nonintegrable Hamiltonian systems. Generally speaking, in the canonical method, nonlinear TPBVPs are linearized with the aid of Hamilton–Jacobi theory and canonical perturbation theory. This cannot be done for non-Hamiltonian systems or for large perturbations.

Because the canonical method exploits iterated linearization to solve TPBVPs, it is important to assure the validity of linearization. Besides, accumulation of error with time due to secular terms restricts the valid time span of the linearization for boundary-value problems. From Sec. II.D, it is clear that the linearization error arises from the omission of the third- and higher-order terms of the perturbation Hamiltonian in Eq. (11). However, the differential operator in Eq. (10) adds complexity to the relation between the error of the perturbation Hamiltonian and the error of $\boldsymbol{r}$. Because of these difficulties, practical error and valid time span estimates have not been worked out in this Note, although they are important in practical applications.

III. Canonical Solution of Two-Point Boundary-Value Problems for Satellite Formation Flying

In this section, the canonical method presented in Sec. II is applied to TPBVPs for the relative motion of satellite formation flying. First, the linearized canonical solution of satellite relative motion presented by Kasdin et al. [7] and Koleman et al. [8] is introduced as the basis for this application. Then, the analytic canonical solution of unperturbed TPBVPs for the relative motion of satellite formation flying is developed. After that, nonlinear effects and the oblateness of the Earth are taken into account.

A. Hamiltonian Modeling and Canonical Solution of Satellite Relative Motion

Under the assumption of a two-body problem and circular reference orbit, by expanding the gravitational potential of the central body using Legendre polynomials and expressing the kinetic energy using relative velocity in the reference frame, Kasdin et al. developed the Lagrangian and Hamiltonian of the relative motion of spacecraft formation flying [7]. The Hamiltonian of a unit mass is as follows [7,8]:

$$
\begin{aligned}
H(\boldsymbol{q},\boldsymbol{p}) &= H^{(0)}(\boldsymbol{q},\boldsymbol{p}) + H^{(1)}(\boldsymbol{q},\boldsymbol{p})\\
H^{(0)}(\boldsymbol{q},\boldsymbol{p}) &= \tfrac{1}{2}(p_x+y)^2 + \tfrac{1}{2}(p_y-x-1)^2 + \tfrac{1}{2}p_z^2 - \tfrac{3}{2} - \tfrac{3}{2}x^2 + \tfrac{1}{2}z^2\\
H^{(1)}(\boldsymbol{q},\boldsymbol{p}) &= -\sum_{k=3}^{\infty} P_k\left(-x/\sqrt{x^2+y^2+z^2}\right)\left(\sqrt{x^2+y^2+z^2}\right)^k
\end{aligned}
\tag{16}
$$

where the generalized coordinates $\boldsymbol{q}=(x,y,z)$, the generalized momenta $\boldsymbol{p}=(p_x,p_y,p_z)=(\dot{x}-y,\dot{y}+x+1,\dot{z})$, $H^{(0)}(\boldsymbol{q},\boldsymbol{p})$ is the linear part of two-body relative Hamiltonian $H(\boldsymbol{q},\boldsymbol{p})$, $H^{(1)}(\boldsymbol{q},\boldsymbol{p})$ is the high-order perturbation part, and $P_k(\cdot)$ are the Legendre polynomials. Here, positions and velocities are represented in a rotating Cartesian Euler–Hill frame [7,8].

Note that, in Eq. (16), all distances are normalized by a, the radius of the circular reference orbit; all rates are normalized by the mean motion $n=\sqrt{\mu/a^3}$, where μ is the gravitational constant.

Then, unperturbed relative motion with the Hamiltonian $H^{(0)}(\boldsymbol{q},\boldsymbol{p})$ was solved by separation of variables, and the linearized relative motion was obtained [7,8]:

$$
\begin{aligned}
x(t) &= 2\alpha_3 + \sqrt{2\alpha_1+3\alpha_3^2}\sin(t+\beta_1)\\
y(t) &= \beta_3 - 3\alpha_3(t+\beta_1) + 2\sqrt{2\alpha_1+3\alpha_3^2}\cos(t+\beta_1)\\
z(t) &= \sqrt{2\alpha_2}\sin(t+\beta_2)
\end{aligned}
\tag{17}
$$

Although Eq. (17) is very similar to the solution of the well-known Clohessy–Wiltshire equations (i.e., linearized equations of satellite relative motion derived by Clohessy and Wiltshire, see [9]), it is important because it is a canonical representation of the relative motion. When perturbations exist, $\boldsymbol{Q}=(\beta_1,\beta_2,\beta_3)$ and $\boldsymbol{P}=(\alpha_1,\alpha_2,\alpha_3)$ are no longer constants, but the form of the solution is still accurate, which is critical for solving TPBVPs with perturbations, as stated in Sec. II.

Gurfil and Kasdin also offered a canonical solution for Hill's three-body problem in [10], which can be used to solve TPBVPs for the three-body problem combined with the canonical method provided in this Note.

B. Solving Two-Point Boundary-Value Problems for Satellite Formation Flying Without Perturbations

Let $\boldsymbol{q}_0=(x_0,y_0,z_0)$ and $\boldsymbol{q}_f=(x_f,y_f,z_f)$ represent the generalized coordinates at $t_0=0$ and t_f, respectively. Then, from Eq. (17),

$$
\begin{aligned}
z_0 &= \sqrt{2\alpha_2}\sin\beta_2, \qquad z_f = \sqrt{2\alpha_2}(\sin t_f\cos\beta_2 + \cos t_f\sin\beta_2)\\
x_0 &= 2\alpha_3 + \sqrt{2\alpha_1+3\alpha_3^2}\sin\beta_1\\
x_f &= 2\alpha_3 + \sqrt{2\alpha_1+3\alpha_3^2}(\sin t_f\cos\beta_1 + \cos t_f\sin\beta_1)\\
y_0 &= \beta_3 - 3\alpha_3\beta_1 + 2\sqrt{2\alpha_1+3\alpha_3^2}\cos\beta_1\\
y_f &= \beta_3 - 3\alpha_3(t_f+\beta_1) + 2\sqrt{2\alpha_1+3\alpha_3^2}(\cos t_f\cos\beta_1\\
&\quad - \sin t_f\sin\beta_1)
\end{aligned}
\tag{18}
$$

the analytic solution of canonical constants $(\boldsymbol{Q},\boldsymbol{P})$ are obtained:

$$
\begin{aligned}
\alpha_2 &= \frac{1}{2}\left[\left(\frac{z_f - z_0\cos t_f}{\sin t_f}\right)^2 + z_0^2\right], \qquad \sin\beta_2 = z_0/\sqrt{2\alpha_2}\\
\cos\beta_2 &= (z_f - z_0\cos t_f)/\sqrt{2\alpha_2}\sin t_f\\
\alpha_1' &= \sqrt{(x_0-2\alpha_3)^2 + [x_f - 2\alpha_3 - (x_0-2\alpha_3)\cos t_f]^2/\sin^2 t_f}\\
\alpha_1 &= (\alpha_1'^2 - 3\alpha_3^2)/2, \qquad \sin\beta_1 = (x_0-2\alpha_3)/\alpha_1'\\
\cos\beta_1 &= [x_f - 2\alpha_3 - (x_0-2\alpha_3)\cos t_f]/\alpha_1'\sin t_f\\
\alpha_3 &= \frac{(y_0-y_f)\sin t_f - 2(x_0+x_f)(1-\cos t_f)}{3t_f\sin t_f - 8(1-\cos t_f)}\\
\beta_3 &= y_0 + 3\alpha_3\beta_1 - 2[x_f - 2\alpha_3 - (x_0-2\alpha_3)\cos t_f]/\sin t_f
\end{aligned}
\tag{19}
$$

Having $(\boldsymbol{Q},\boldsymbol{P})$, the expressions of $\dot{x}$, $\dot{y}$, and $\dot{z}$ are obtained from Eq. (17); then, $(\dot{x}_0,\dot{y}_0,\dot{z}_0)$, $(\dot{x}_f,\dot{y}_f,\dot{z}_f)$, $\boldsymbol{p}_0$, and $\boldsymbol{p}_f$ can be computed. Thus, the TPBVPs for satellite relative motion without perturbations are solved.

Note that the previous derivation is a little different from that of Sec. II.A. Hamilton's principal function $S(\boldsymbol{q},\boldsymbol{P},t)$ is not used explicitly. However, Eqs. (3) and (4) are equivalent. Because the solution of relative motion given in Eq. (17) is already obtained, it is used instead.

C. Two-Point Boundary-Value Problems for Satellite Formation Flying with Nonlinear Perturbations

For a two-body problem, $H^{(1)}(\boldsymbol{q}, \boldsymbol{P})$ in Eq. (16) includes all high-order (i.e., nonlinear) perturbations. As many high-order terms as possible can be included according to the precision requirements when solving TPBVPs. However, for simplicity and satisfactory precision, only the third- and fourth-order terms in the perturbation Hamiltonian are consider here.

The truncated perturbation Hamiltonian is [7,8]

$$H^{(1)}(\boldsymbol{q}, \boldsymbol{p}) = x^3 - \tfrac{3}{2}xy^2 - \tfrac{3}{2}xz^2 - x^4 + 3x^2y^2 + 3x^2z^2 - \tfrac{3}{8}y^4 - \tfrac{3}{4}y^2z^2 - \tfrac{3}{8}z^4 \tag{20}$$

Substituting the coordinates with Eq. (17), the previous perturbation Hamiltonian is expressed in canonical constants, $(\boldsymbol{Q}, \boldsymbol{P})$. For conciseness, the full expression of $H^{(1)}(\boldsymbol{Q}, \boldsymbol{P})$ is not presented here.

Given $H^{(1)}(\boldsymbol{Q}, \boldsymbol{P})$, the perturbation differential equations of $\boldsymbol{Q}(t), \boldsymbol{P}(t)$ can be obtained from Eq. (6). The boundary conditions for $\boldsymbol{Q}(t), \boldsymbol{P}(t)$ are

$$\boldsymbol{q}_0 = \boldsymbol{q}(\boldsymbol{Q}_0, \boldsymbol{P}_0, t_0) \qquad \boldsymbol{q}_f = \boldsymbol{q}(\boldsymbol{Q}_f, \boldsymbol{P}_f, t_f) \tag{21}$$

which is equivalent to Eq. (7). Equation (21) is used instead, because the solution of the relative motion $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q}, \boldsymbol{P}, t)$ is already obtained from Eq. (17).

Thus, TPBVPs with nonlinear perturbations can be solved using the approach presented in Sec. II.

D. Two-Point Boundary-Value Problems for Satellite Formation with the Earth's Nonspherical Perturbations

Oblateness of the Earth causes the most significant perturbation to Earth satellites, especially for low Earth orbits. Treating the Earth as axially symmetric is a good approximation. Thus, the gravitational potential includes zonal harmonics [5]:

$$U = -\frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty}\left(\frac{R_e}{r}\right)^k J_k P_k(\sin\varphi)\right] \tag{22}$$

where R_e is the mean equatorial radius of the Earth, J_k are the constants of the zonal harmonics, φ is the latitude angle, r is the distance from the gravitational center of the Earth, and $P_k(\cdot)$ is the Legendre polynomial.

To simplify the statement, only the most important part of the gravitational zonal harmonics, the J_2 term, is considered. Thus, the following is added to the two-body gravitational potential:

$$U_2 = \frac{\mu}{r}\left(\frac{R_e}{r}\right)^2 \frac{J_2}{2}\left[3\left(\frac{Z}{r}\right)^2 - 1\right] \tag{23}$$

where Z is the normal deflection in an inertial geocentric-equatorial reference frame.

According to the transformation relationship of coordinates from the circular reference frame to the inertial geocentric-equatorial reference frame,

$$Z = a\sin u \sin i + x \sin u \sin i + y \cos u \sin i + z \cos i \tag{24}$$

where i and u are the inclination and argument of latitude of the reference orbit, respectively.

Substituted into Eq. (23), the J_2 gravitational potential perturbation becomes

$$U_2 = \frac{J_2}{2}\left(\frac{R_e}{a}\right)^2\left(\frac{1}{r}\right)^3\left[3(\sin u \sin i + x \sin u \sin i + y \cos u \sin i + z \cos i)^2\left(\frac{1}{r}\right)^2 - 1\right] \tag{25}$$

Expanding $\frac{1}{r}$ using the Legendre polynomials [7,8],

$$\frac{1}{r} = \sum_{k=0}^{\infty} P_k(-x/\sqrt{x^2+y^2+z^2})(\sqrt{x^2+y^2+z^2})^k \tag{26}$$

Substituting Eq. (26) into Eq. (25), the perturbation potential is expressed in the generalized coordinates of relative motion, $\boldsymbol{q}$. After normalization, $U_2(\boldsymbol{q})$ is just the J_2 perturbations Hamiltonian, $H^{(2)}(\boldsymbol{q}, \boldsymbol{p})$. The expression is not presented here for conciseness. In the following case study, it is enough to only keep those terms with an order of two or lower of the generalized coordinates in $H^{(2)}(\boldsymbol{q}, \boldsymbol{p})$.

Then, the problem can be solved following the method presented in Sec. III.C.

IV. Solving Two-Point Boundary-Value Problems of Satellite Relative Motion: Case Studies

In this section, TPBVPs of satellite relative motion are solved with the proposed canonical approach. Example problems of satellite relative motion in the central-potential field of the Earth and a J_2-perturbed potential field are presented.

A. Solving Two-Point Boundary-Value Problems of Satellite Relative Motion in Central-Potential Field

Under the two-body assumption, in the central-potential field of the Earth, satellites move in Keplerian orbits. Suppose the chief (reference) satellite is in a circular orbit with an altitude of 800 km and inclination of 30°. Given the coordinates of the deputy satellite relative to the chief in the Euler–Hill reference frame at initial time $t_0 = 0$ and final time $t_f = 3000$ s, the problem is to find the velocity boundary values. For comparison, two groups of data are used for the boundary coordinates. For the first group of data, the initial and final coordinates are $\boldsymbol{q}_0 = (0, -1000, 0)$ and

$$\boldsymbol{q}_f = (749.415349, -559.916544, 279.958272)$$

respectively. For the second group of data, the values are 10 times larger than the first.

Solutions of the two TPBVPs are listed in Tables 1 and 2, respectively. For both problems, two kinds of solutions are provided: one solves the problems without considering the nonlinear part of the Hamiltonian, as in Sec. III.B, and the other solves the problems considering the third- and fourth-order nonlinear perturbation parts of the Hamiltonian, as in Sec. III.C. Thus, the two solutions both use approximate models, which are different from the full two-body problem. To compute the solution error, the final theoretical coordinates and velocities are determined from the given initial coordinates and the initial velocity solutions according to the Kepler motion law (i.e., Keplerian orbit). The differences between the solved final states and the theoretical final states represent the precision of the solutions. The differences are stated as errors and listed in the tables. They are presented in the Earth-centric inertial (ECI) frame.

From Tables 1 and 2, it is evident that the errors can be very significant without considering the nonlinear terms. When considering the third and fourth nonlinear terms as perturbations, and using the canonical approach, the solutions are greatly improved after only a few iterations.

Because the main computation in each iteration is the numerical integration of seven linearized perturbation differential equations to obtain a special solution of the nonhomogeneous equations and the fundamental system of solutions of the homogeneous equations [6], the computation cost is much smaller than that of other approaches, such as the shooting method and parameterized optimization.

Comparing solutions of the second problem with those of the first, the errors appear to be larger because the relative distance is 10 times larger for the second problem; thus, the nonlinear effect is much greater. It is shown that, for the scale of the second problem, considering only the third and fourth nonlinear terms in the perturbing Hamiltonian is not enough for high precision, whereas for the canonical method presented in this Note, it is very simple to add

Table 1 Solutions of the TPBVP with the first group of boundary conditions under the two-body assumption

		Boundary velocities, m/s		Errors	
	Iteration	Initial	Final	Position, m	Velocity, m/s
Solving without perturbations	——	−0.572408	0.582991	27.854485	0.071116
		0.198427	−1.357553	60.383400	0.027814
		10.684076	−10.680122	34.273985	0.017406
Solving with third- and fourth-order nonlinear perturbations included	First	−0.572491	0.583388	0.789120	0.001643
		0.190890	−1.365490	1.390303	0.000409
		10.707304	−10.702220	0.803644	0.000238
	Second	−0.572402	0.583298	6.850809e − 5	1.002158e − 7
		0.190673	−1.365271	7.438432e − 5	5.989841e − 8
		10.707361	−10.702276	3.957274e − 5	3.670448e − 8
	Third	−0.572402	0.583298	5.911477e − 5	1.181236e − 7
		0.190673	−1.365271	0.000111	5.799939e − 8
		10.707361	−10.702276	6.095476e − 5	3.558125e − 8

additional high-order terms of the nonlinear perturbations if higher precision is needed for an application with a larger scale.

B. Solving Two-Point Boundary-Value Problems of Satellite Relative Motion in J_2-Perturbed Potential Field

For simplicity, it is assumed that there is a virtual chief satellite moving in a Keplerian orbit with the same orbital characteristics as that described in Sec. IV.A. Thus, the virtual chief is not affected by perturbations. It provides a reference frame, as in Sec. IV.A, while the deputy satellite moves in the J_2-perturbed potential field and stays close to the virtual chief. Coordinates of the deputy satellite relative to the chief in the Euler–Hill reference frame are $\boldsymbol{q}_0 = (0, -1000, 0)$ at the initial time $t_0 = 0$ and

$$\boldsymbol{q}_f = (749.415349, -559.916544, 279.958272)$$

at the final time $t_f = 300$ s. The problem is to find the velocity boundary values.

Two kinds of solutions are provided: one solves the problems without considering perturbation of the Hamiltonian, as in Sec. III.B, while the other solves the problems with J_2 perturbations by keeping terms of the generalized coordinates in $H^{(2)}(\boldsymbol{q}, \boldsymbol{p})$ up to an order of two, as stated in Sec. III.D. Solutions are listed in Table 3. Errors listed in Table 3 are differences between the final states of the solutions and theoretical final states, as in Tables 1 and 2. However, the theoretical final states here are computed by numerical integration of the J_2-perturbed orbital dynamics, with the initial coordinates and the initial velocity solutions as initial states. The errors represent the precision of the solutions for an ideal J_2-perturbed potential field. They are presented in the ECI frame.

Similarly, it is shown that the errors are large without considering perturbations. After only a few iterations using the canonical approach, the solutions are greatly improved and high precision is achieved.

In the previous example, the final time is set at $t_f = 300$ s, which is comparatively smaller than the examples in Sec. IV.A. Here, for a longer time span, the iterations diverge. If the orbit altitude of the virtual chief is raised, the time span that guarantees convergence increases. Some examples for higher reference orbits are listed in Table 4.

Convergence is not guaranteed for a large time span, because the J_2 perturbations are significant, especially for low Earth orbits. Therefore, the accumulation of the error with time destroys the validity of the linearization for a larger time span. As the orbit altitude increases, the effect of the J_2 perturbations decreases; thus, the convergence range extends, as shown in Table 4. If the main part of

Table 2 Solutions of the TPBVP with the second group of boundary conditions under the two-body assumption

		Boundary velocities, m/s		Errors	
	Iteration	Initial	Final	Position, m	Velocity, m/s
Solving without perturbations	——	−5.724076	5.829912	2775.906324	7.093388
		1.984269	−13.575527	5992.908175	2.764775
		106.840759	−106.801222	3504.373987	1.779565
Solving with third- and fourth-order nonlinear perturbations included	First	−5.837338	5.942273	748.273415	1.471740
		1.382781	−14.607914	1211.189388	0.360915
		109.256807	−109.101857	724.448512	0.222676
	Second	−5.722486	5.859663	2.829686	0.014395
		1.178881	−14.376545	12.789572	0.003340
		109.211810	−109.057104	7.692480	0.001968
	Third	−5.724394	5.860647	0.626023	0.001488
		1.177878	−14.377644	1.273190	0.000616
		109.211887	−109.057177	0.700938	0.000395
	Fourth	−5.724394	5.860648	0.623334	0.001480
		1.177878	−14.377645	1.267109	0.000614
		109.211887	−109.057177	0.697315	0.000394

Table 3 Solutions of the TPBVP under J_2 perturbations

	Iteration	Boundary velocities, m/s Initial	Boundary velocities, m/s Final	Errors Position, m	Errors Velocity, m/s
Solving without perturbations	——	1.854384	3.101270	440.951631	2.904276
		2.180108	0.624129	39.994056	0.399598
		0.948452	0.902826	68.938997	0.686403
Solving with J_2 perturbations to the second-order of generalized coordinates included	First	3.511111	1.469589	69.547780	0.014220
		1.237816	−0.826443	259.915893	0.276459
		0.825881	0.360774	237.672599	0.195275
	Second	3.304271	1.507243	7.483007	0.0799423
		2.110079	0.251859	71.505723	0.1086452
		1.059491	0.607361	49.280387	0.0703355
	Third	3.282250	1.677086	1.257902	0.004358
		2.375766	0.557563	7.795755	0.034649
		1.088354	0.640919	2.661719	0.020571
	Fourth	3.278297	1.688843	0.091898	0.000339
		2.401627	0.621043	0.386129	0.000808
		1.082951	0.636252	0.225372	0.000483
	Fifth	3.278006	1.689624	0.005147	1.861281e − 5
		2.403127	0.623320	0.004174	3.645727e − 5
		1.082961	0.636265	0.001404	8.580677e − 6

Table 4 Solutions of the TPBVP under J_2 perturbations for higher reference orbits

Altitude, km	Final time, s	Number of iterations	Boundary velocities, m/s Initial	Boundary velocities, m/s Final	Errors Position, m	Errors Velocity, m/s
12,000	1200	6	0.637081	0.600645	0.003334	3.372979e − 6
			0.607566	0.118145	0.001928	4.895859e − 6
			0.261563	0.177429	0.000592	1.137409e − 6
20,000	3000	3	0.223732	0.268981	0.005273	2.064628e − 6
			0.259585	0.021823	0.002636	2.557477e − 6
			0.106980	0.066469	0.003389	5.909703e − 7

the J_2-perturbation Hamiltonian could be separated to enrich the separable Hamiltonian, $H^{(0)}(\boldsymbol{q}, \boldsymbol{p})$, a larger range of convergence would probably be achieved.

V. Conclusions

A canonical method is presented for TPBVPs of Hamiltonian systems, in which the Hamiltonian function can be divided into a major analytical integrable part and a minor perturbation part. For the major part of the Hamiltonian, Hamilton's principal function and canonical transformation are used to find the canonical constants. Then, taking the perturbations into account, the original TPBVPs are transformed to new TPBVPs expressed in terms of the canonical constants, which are now variable and satisfy the canonical perturbation equations. Assuming that the minor perturbation Hamiltonian is relatively small, the new TPBVPs are solved by linearization. By updating the canonical constants as the new reference point for linearization, the linearized solving process is iterated to achieve high-order precision. The canonical method is shown more specifically for the relative motion of satellite formation flying, with both nonlinear effect and J_2 perturbations considered. Case studies are presented, and accurate solutions are obtained with a small computational cost.

The approach presented in this Note relies on the Hamilton–Jacobi modeling and solutions. As the number of terms in the Hamiltonian that can be resolved by separation of variables increases, the convergence performance improves.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Number 10802094). The authors thank Sandor M. Veres of the University of Southampton for reading this work and giving suggestions.

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